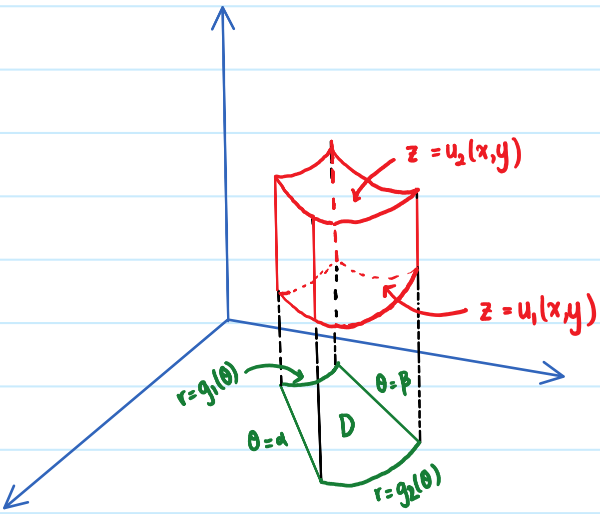
Triple Integrals w/ cylindrical coordinates

Suppose E is a Type 1 region whose projection D onto the xy-plane is described in polar coordinates.

Spse f is continuous and

$E = \{(x,y,z) \mid x \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$ and D is given in polar coordinates by

$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$



We Know that,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

Now notice that the double integral is in polar coordinates.

Therefore we obtain the formula for triple integration in cylindrical coordinates.

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(x,y)}^{u_2(x,y)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta .}$$

Ex Evaluate $\iiint_E x+y+z\, dV$ where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.

Cylindrical coordinates $z=4-r^2$

- The paraboloid $z=4-(x^2+y^2)$ intersects the xy-plane in the circle $x^2+y^2=4 \Rightarrow r=2$.

So, $E=\{(r,\theta,z) \mid 0\le\theta\le\frac{\pi}{2},\ 0\le r\le 2,\ 0\le z\le 4-r^2\}$

Then,

$$\iiint_E (x+y+z)\,dV = \int_0^{\pi/2}\int_0^2\int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 r^2(\cos\theta+\sin\theta)z + \frac{1}{2}rz^2\Big]_{z=0}^{4-r^2} = \int_0^{\pi/2}\int_0^2 [4r^2-r^4](\cos\theta+\sin\theta) + \frac{1}{2}r(4-r^2)^2\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{4r^3}{3} - \frac{1}{5}r^5\right](\cos\theta+\sin\theta) - \frac{1}{12}(4-r^2)^3\Big]_{r=0}^{2} d\theta$$

$$= \int_0^{\pi/2}\left[\frac{64}{15}(\cos\theta+\sin\theta) + \frac{16}{3}\right]d\theta = \left[\frac{64}{15}(\sin\theta-\cos\theta)\right]_0^{\pi/2}$$

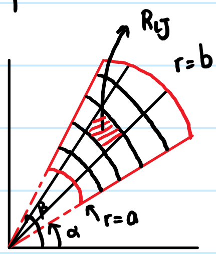
$$= \frac{64}{15}(1-0) + \frac{16}{3}\cdot\frac{\pi}{2} - \frac{64}{15}(0-1) - 0$$

$$= \frac{8\pi}{3} + \frac{128}{15}$$

Volume element in Cylindrical coordinates

So in rectangular coordinates, we Know that volume element $dV = dx\,dy\,dz$.

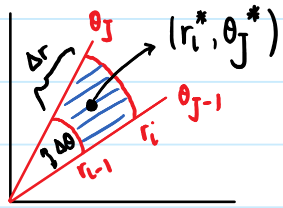
The question is what is the (infinitesimal) volume element in cylindrical coordinates ?



- So divide $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b-a)/m$
- Divide $[\alpha, \beta]$ into n equal subintervals $[\theta_{J-1}, \theta_J]$ of equal width $\Delta\theta = \frac{\beta-\alpha}{n}$.

- Now consider the polar rectangle R_{iJ}.
  and consider the center of the rectangle given by $r_i^* = \frac{1}{2}(r_{i-1}+r_i)$, $\theta_J^* = \frac{1}{2}(\theta_{J-1}+\theta_J)$

What is the area of R_{iJ} ?

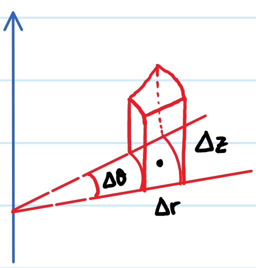


Now we want to find the area of the shaded sector, which we obtain by subtracting area of two sectors.

$$\text{So}\quad A(R_{iJ}) = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\,\Delta\theta = \underbrace{\frac{1}{2}(r_i+r_{i-1})}_{r_i^*}\underbrace{(r_i-r_{i-1})}_{\Delta r}\Delta\theta = \underbrace{r_i^*}\Delta r\,\Delta\theta$$

This is the extra r the shows up in the double integral w/ polar coordinates.

Now what about the volume element ?



Then we see that

$$\Delta V \approx r_i^*\,\Delta r\,\Delta\theta\,\Delta z$$

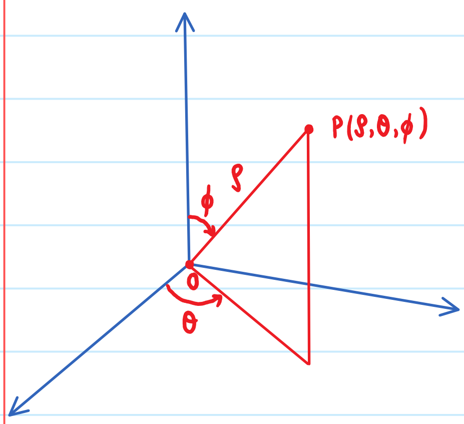
So,

$$dV = r\,dr\,d\theta\,dz$$

15.9 Triple Integral in spherical coordinates

• Spherical coordinates

A point P in 3-dim'l space is represented by an ordered triple (ρ, θ, ϕ), where

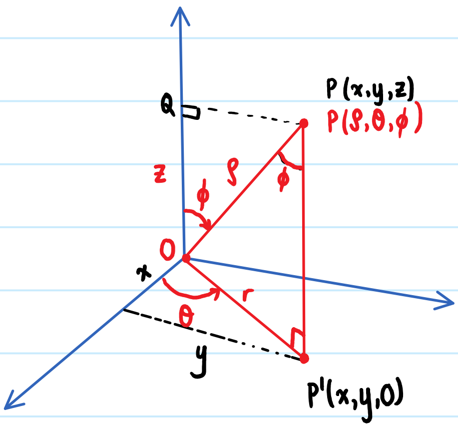


$\rho \equiv$ distance from the origin to P, $(\rho = |OP|)$
$\theta \equiv$ same angle as in cylindrical coordinates. ("polar angle")
$\phi \equiv$ the angle between the z-axis and the line segment $\overrightarrow{OP}$. (azimuth angle).

Note $\rho \geq 0$, $0 \leq \phi \leq \pi$

• Spherical coordinates are useful when there is a symmetry about the origin.

Relationship between rectangular and spherical coordinates



From ΔOPQ, we see that $\cos\phi = \frac{z}{\rho} \Rightarrow z = \rho\cos\phi$

From $\Delta OPP'$, $\sin\phi = \frac{r}{\rho} \Rightarrow r = \rho\sin\phi$

Now on the other hand, we know
$x = r\cos\theta$, $y = r\sin\theta$.

Putting evth together we get

$$\boxed{\begin{array}{l} \bullet\ x = \rho\sin\phi\cos\theta \\ \bullet\ y = \rho\sin\phi\sin\theta \\ \bullet\ z = \rho\cos\phi \end{array}}$$

• From the distance formula, we get

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

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<u>Ex</u> Find the spherical coordinates for the point P whose rectangular coordinate is $(-1, 1, -\sqrt{2})$.

<u>Soln</u> $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+1+2} = 2$

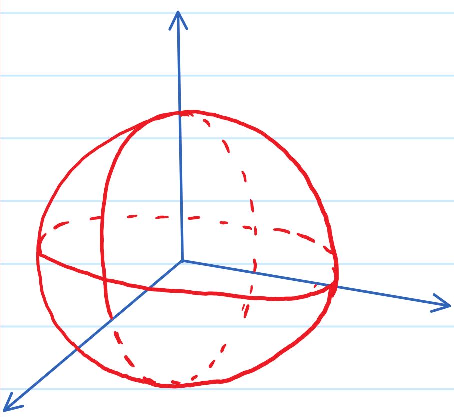
- $\cos\phi = \dfrac{z}{\rho} \Rightarrow \cos\phi = -\dfrac{\sqrt{2}}{2} \Rightarrow \phi = \dfrac{3\pi}{4}$

- $x = \rho\sin\phi\cos\theta \Rightarrow \cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{-1}{2 \cdot \sqrt{2}/2} = \dfrac{-1}{\sqrt{2}}$

At this point $\theta = \dfrac{3\pi}{4}$ or $\dfrac{5\pi}{4}$. So we need to look at y which is >0, meaning $\theta = \dfrac{3\pi}{4}$.

<u>Ex</u> Describe the surface whose equations in spherical coordinates is given by

a) $\rho = c$



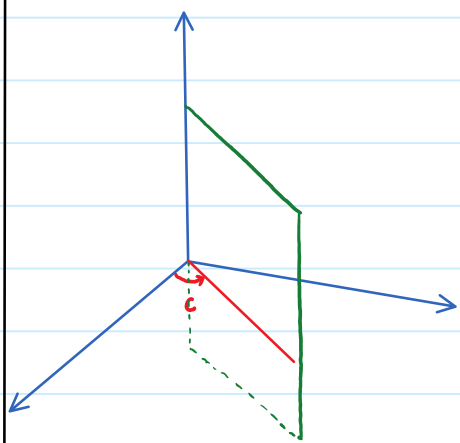
$\rho = c$

$\Rightarrow \sqrt{x^2+y^2+z^2} = c$

$\Rightarrow x^2+y^2+z^2 = c^2$

A sphere of radius c

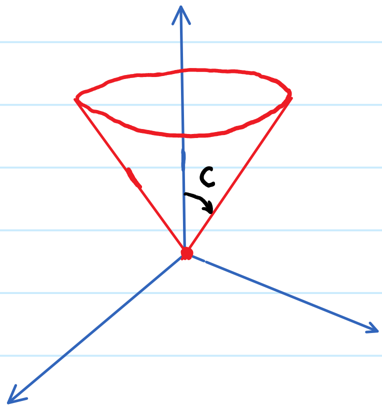
b) $\theta = c$



$\theta = c$

- Now matter how far we go from the origin, it's projection onto the xy plane must make angle θ with the x-axis.

c) $\phi = c$, $(0 < c < \pi/2)$



- No matter how far we move from the origin, the point must have angle c wrt to the z-axis.
- All points on the cone are at the same angle from the z-axis [compare w/ cylindrical]

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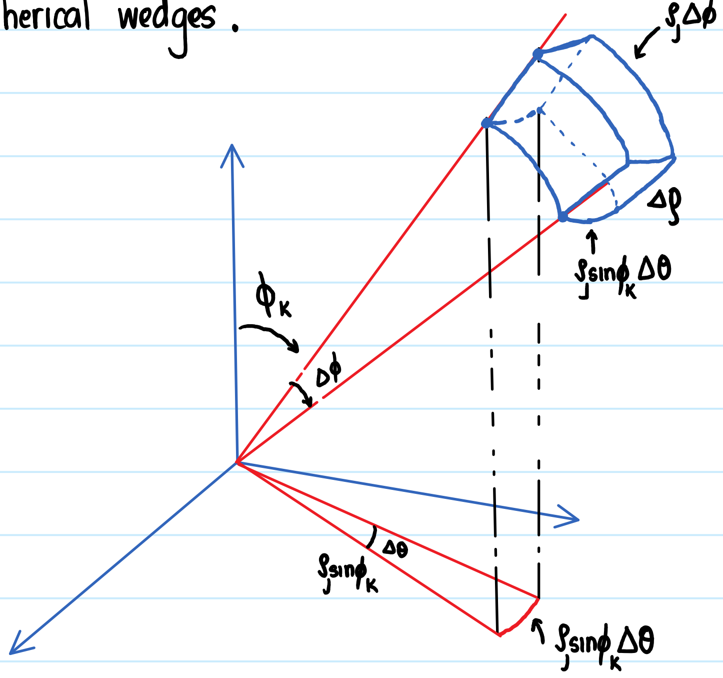
Triple Integrals w/ spherical coordinates

- In spherical coordinate system, the analog of a rectangular box is a <u>spherical wedge</u>

$$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d,\ a \ge 0,\ \beta-\alpha \le 2\pi,\ d-c \le \pi\}$$

- So when dividing the region (rather than using rectangular boxes) we can use spherical wedges.



- Divide E into smaller wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half planes $\theta = \theta_j$ and half cones $\phi = \phi_k$.

- Then E_{ijk} is approximately a rectangular box w/ dimension $\Delta\rho$, $\rho_i \Delta\phi$ and $\rho_i \sin\phi_k \Delta\theta$

$\rho_i \Delta\phi$ ↓ arc of radius ρ_i and angle $\Delta\phi$

$\rho_i \sin\phi_k \Delta\theta$ ↓ arc of radius $\rho_i \sin\phi_k$ and angle $\Delta\theta$.

Then, $\Delta V_{ijk} \approx \rho_i^2 \sin\phi_k \Delta\rho \Delta\theta \Delta\phi$

With a little bit of work, we can show that

$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi$ for some point in E_{ijk}.

Then,

$$\iiint_E f(x,y,z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j,\ \tilde{\rho}_j \sin\tilde{\phi}_k \cos\tilde{\theta}_j,\ \tilde{\rho}_i \cos\tilde{\phi}_k)\, \tilde{\rho}_i^2 \sin\tilde{\phi}_j\, \Delta\rho\Delta\theta\Delta\phi$$

Formula for triple integration in spherical coordinates

$$\iiint_E f(x,y,z)\,dV = \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

where E is the spherical wedge given by :

$$E = \{(\rho,\theta,\phi)\mid 0\le a\le\rho\le b,\ \alpha\le\theta\le\beta,\ c\le\phi\le d\}$$

- Like in the case of rectangular coordinates and cylindrical coordinates, the formula can be extended to more general spherical regions such as

$$E = \{(\rho,\theta,\phi)\mid \alpha\le\theta\le\beta,\ c\le\phi\le d,\ g_1(\theta,\phi)\le\rho\le g_2(\theta,\phi)\}$$

Ex Use spherical coordinates to find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$.

Soln In spherical coordinates, $x^2+y^2+z^2=4$ is given by $\rho=2$. and the cone $z=\sqrt{x^2+y^2}$ can be rewritten as

$$\rho\cos\phi = \rho\sin\phi \Rightarrow \cos\phi=\sin\phi \Rightarrow \phi = \pi/4$$

Thus the solid in question is given by

$$\{(\rho,\theta,\phi)\mid 0\le\rho\le 2,\ 0\le\theta\le 2\pi,\ \underline{\tfrac{\pi}{4}\le\phi\le\tfrac{\pi}{2}}\}$$

Ask students to answer.

Then,

$$V(E) = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^2\,d\rho = \frac{\sqrt{2}}{2}\cdot 2\pi\cdot\frac{8}{3} = \frac{8\sqrt{2}\,\pi}{3}$$

<u>Ex</u> Evaluate $\iiint\limits_E x e^{x^2+y^2+z^2}\, dV$, where E is the portion of the unit ball $x^2+y^2+z^2 \le 1$ that lies in the first octant.

$$E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le \phi \le \tfrac{\pi}{2}\}$$

Then, $\iiint\limits_E x e^{x^2+y^2+z^2}\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho \sin\phi \cos\theta\, \rho^2 \sin\phi\ d\rho\, d\theta\, d\phi$

$$= \int_0^{\pi/2} \sin^2\phi\, d\phi \int_0^{\pi/2} \cos\theta\, d\theta \int_0^1 \rho^3 e^{\rho^2}\, d\rho$$

$$= \left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^{\pi/2} \left[\sin\theta\right]_0^{\pi/2} \cdot \left(\left[\tfrac{1}{2}\rho^2 e^{\rho^2}\right]_0^1 - \int_0^1 \rho e^{\rho^2}\, d\rho\right)$$

$$= \left[\tfrac{\pi}{4}\right] [1] \left[\tfrac{1}{2}\rho^2 e^{\rho^2} - \tfrac{1}{2} e^{\rho^2}\right]_{\rho=0}^{1}$$

$$= \tfrac{\pi}{4} \cdot 1 \cdot \left[\tfrac{1}{2}e - \tfrac{1}{2}e - \left(0 - \tfrac{1}{2}\right)\right]$$

$$= \tfrac{\pi}{8}$$

15.10 Change of variables in Multiple Integrals

• In 1-dim'l calculus we often use a change of variables to simplify integrals (u-substitution)

$$\int_c^d f(g(u)).g'(u) = \int_a^b f(x)\,dx \text{ , where } x=g(u),\ a=g(c) \text{ and } b=g(d).$$

Another method of writing the above formula is

$$\int_a^b f(x)\,dx = \int_c^d f(x,u)\frac{dx}{du}\,du$$

• Want to extend this to double integrals.

We have already seen an example of this : changing to polar coordinates.

We introduce new variables r and θ, which are related to original variables x & y via the eqns : $x = r\cos\theta$ and $y = r\sin\theta$, and the change of variables formula is :

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

where $S \equiv$ region in $r\theta$-plane that corresponds to R in the xy-plane.

• We want to consider a more general change of variable.

Consider a transformation T from the uv-plane to the xy-plane :

$$T(u,v) = (x,y)$$

where x and y are related to u and v by the equation

$$x = g(u,v),\ y = h(u,v). \quad [\,T(u,v) = (g(u,v), h(u,v))\,]$$

or sometimes written as $x = x(u,v)$, $y = y(u,v)$.

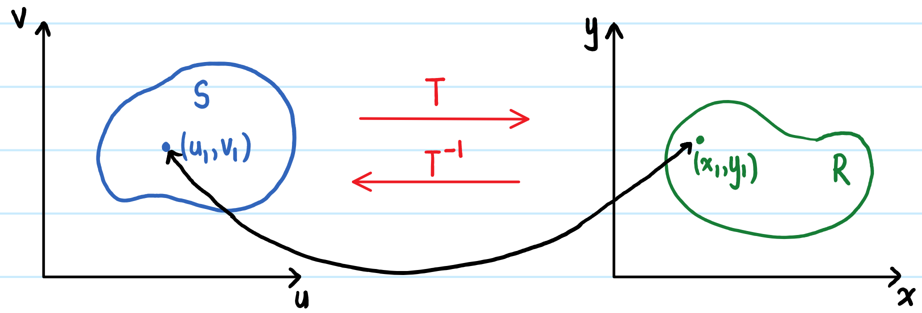
Assume T is a C^1 transformation i.e. g and h have continuous first-order partial derivatives.

- T is just a vector valued function whose domain and range are subsets of $\mathbb{R}^2$.

Notation

- If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the image of (u_1, v_1).

- T is a 1-1 function, if $(u_1, v_1) \neq (u_2, v_2)$ implies $T(u_1, v_1) \neq T(u_2, v_2)$ i.e. no two points have the same image.



- T transforms S (region in the uv-plane) into a region R in the xy-plane called the image of S, consisting of the images of all points in S.

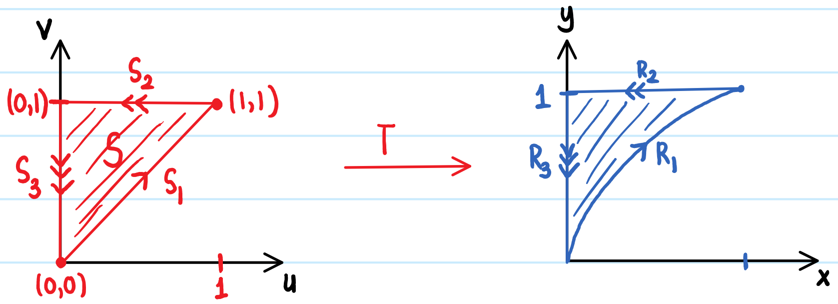
- If T is a 1-1 function, then T^{-1} is a map from the xy-plane and uv-plane and it may be possible to solve $x = g(u,v)$ and $y = h(u,v)$ for u, v in terms of x and y:

$$u = G(x,y), \quad v = H(x,y).$$

- Example Let T be a transformation defined by

$$x = u^2, \quad y = v.$$

Find the image of the triangular region S with vertices $(0,0)$, $(1,1)$ and $(0,1)$.



S_1 : $\{(u,v) \mid 0 \le u \le 1\}$

Then , $y = u = v$ and $x = u^2 = y^2$.
Since, $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$.

S_2 : $\{(u,v) \mid 0 \le u \le 1 , v = 1\}$

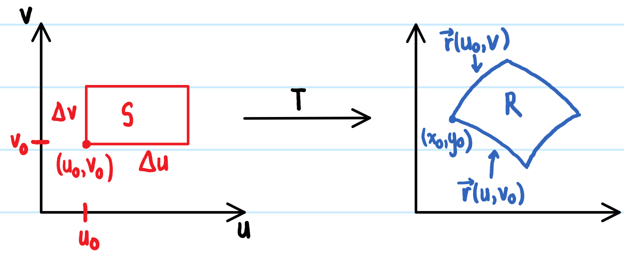
Then $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$.
The image is the line segment $y = 1$, $0 \le x \le 1$.

S_3 : $\{(u,v) \mid u = 0, 0 \le v \le 1\}$

Then $x = u^2 = 0$ and $y = v \Rightarrow 0 \le y \le 1$.
The image is the segment $x = 0$, $0 \le y \le 1$.

Thus the image of S is the region R in the first quandrant bounded by the parabola $x = y^2$, the y-axis and the line $y = 1$.

How does change of variable affects a double integral :



Start w/ a small rectangle in the uv-plane whose left corner point is (u_0,v_0) and whose dimension are Δu and Δv.

Then images of S is a region R in the xy-plane, one of whose boundary points is $(x_0,y_0) = T(u_0,v_0)$.

The vector $\vec{r}(u,v) = g(u,v)\hat{i} + h(u,v)\hat{j}$ is the position vector of the image of the point (u,v).

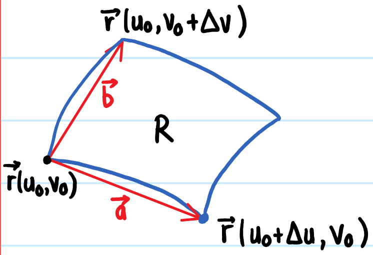
The tangent vector at (x_0,y_0) to this image curve is

$$r_u = g_u(u_0,v_0)\hat{i} + h_u(u_0,v_0)\hat{j} = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j}$$

Similarly, the tangent vector at (x_0,y_0) to the image curve on the left side of S $(u=u_0)$ is

$$r_v = g_v(u_0,v_0)\hat{i} + h_v(u_0,v_0)\hat{j} = \frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}$$

Then we can approximate the image region $R = T(S)$ by a parallelogram by the secant vectors



But as $\Delta u, \Delta v \to 0$, $\vec{a} = \vec{r}_u(u_0,v_0)$ and $\vec{b} = \vec{r}_v(u_0,v_0)$

and so, $\vec{a} \approx \Delta u\, \vec{r}_u$

$\vec{b} \approx \Delta v \cdot \vec{r}_v$

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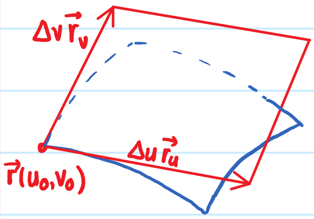
- We can approximate R by a parallelogram, whose area is



$$|\Delta u \vec{r}_u \times \Delta v \vec{r}_v| = |r_u \times r_v| \Delta u \Delta v$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k}$$

<u>DEF</u> The Jacobian of a transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Then the approximate area ΔA of R is :

$$\Delta A = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot \Delta u \Delta v.$$

Now divide region S in the uv-plane into rectangles S_{ij} and $R_{ij} = T(S_{ij})$

Then, $\iint_E f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$

Change of variables in double integrals

Spse T is a <u>C^1-transformation</u>, whose <u>Jacobian is non-zero</u>, and that maps the region S in the uv-plane onto R in the xy-plane.

Spse f is continuous on R and R and S are Type I or type II regions.

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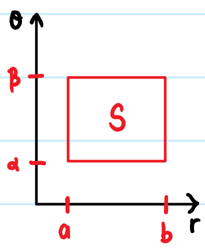
Spse T is 1-1, except perhaps on the boundary of S.

Then, $$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

When you change integral in x and y to u and v by writting x and y in terms of u and v and writing $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.

<u>Ex 1</u> <u>Polar coordinates</u>

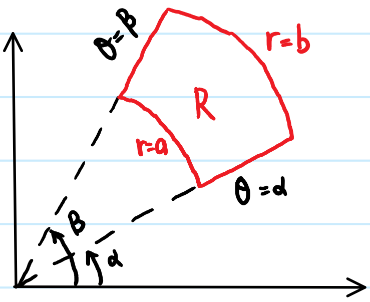


$\xrightarrow{T}$

$T(r,\theta) = (x,y)$

$x = r\cos\theta$

$y = r\sin\theta$



Then $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0.$$

Then, $$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$
$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

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Triple Integrals

- Similar change of variables formula for triple integrals.

Spse T is a linear transformation that maps a region S in uvw-space onto a region R in the xyz-space by means of the equation

$$x = g(u,v,w) \quad , \; y = h(u,v,w) \quad \text{and} \quad z = K(u,v,w).$$

Then the Jacobian of T is a 3x3 determinant :

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Then under similar conditions (as the double integral case).

$$\boxed{\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw}$$